CS355, Spring 25

The PLONK
Polynomial Interactive Oracle Proof (PIOP)

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Review: a General Paradigm for a Modern SNARK

$\mathcal{F}$-functional commitment scheme

(cryptographic object)

$\mathcal{F}$-interactive oracle proof ($\mathcal{F}$ – **IOP**)

(info. theoretic object)

SNARK for general circuits

A special case: Polynomial IOP (PIOP)

univariate polynomial commitment scheme (**PCS**)

Poly Interactive Oracle Proof (**PIOP**)

SNARK for general circuits

Polynomial IOPs (PIOPs)

Setup(C) → public parameters pp and $(vp,$ $f_{-1}, \ldots, f_{-s} \in \mathbb{F}_p^{(\leq d)}[X]$ $)$

Prover $P(pp, \mathbf{x}, \mathbf{w})$

Verifier $V(vp, \mathbf{x})$

oracle: $f_0 \in \mathbb{F}_p^{(\leq d)}[X]$ →

$r_1 \leftarrow \mathbb{F}_p$

← r_1

⋮

← r_{t-1}

$r_{t-1} \leftarrow \mathbb{F}_p$

oracle: $f_t \in \mathbb{F}_p^{(\leq d)}[X]$ →

$$\text{verify}^{f_{-s}, \ldots, f_t}(\mathbf{x}, r_1, , \ldots, r_{t-1}) \rightarrow 0/1$$

Verifier is assured that all oracles are in $\mathbb{F}_p^{(\leq d)}[X]$

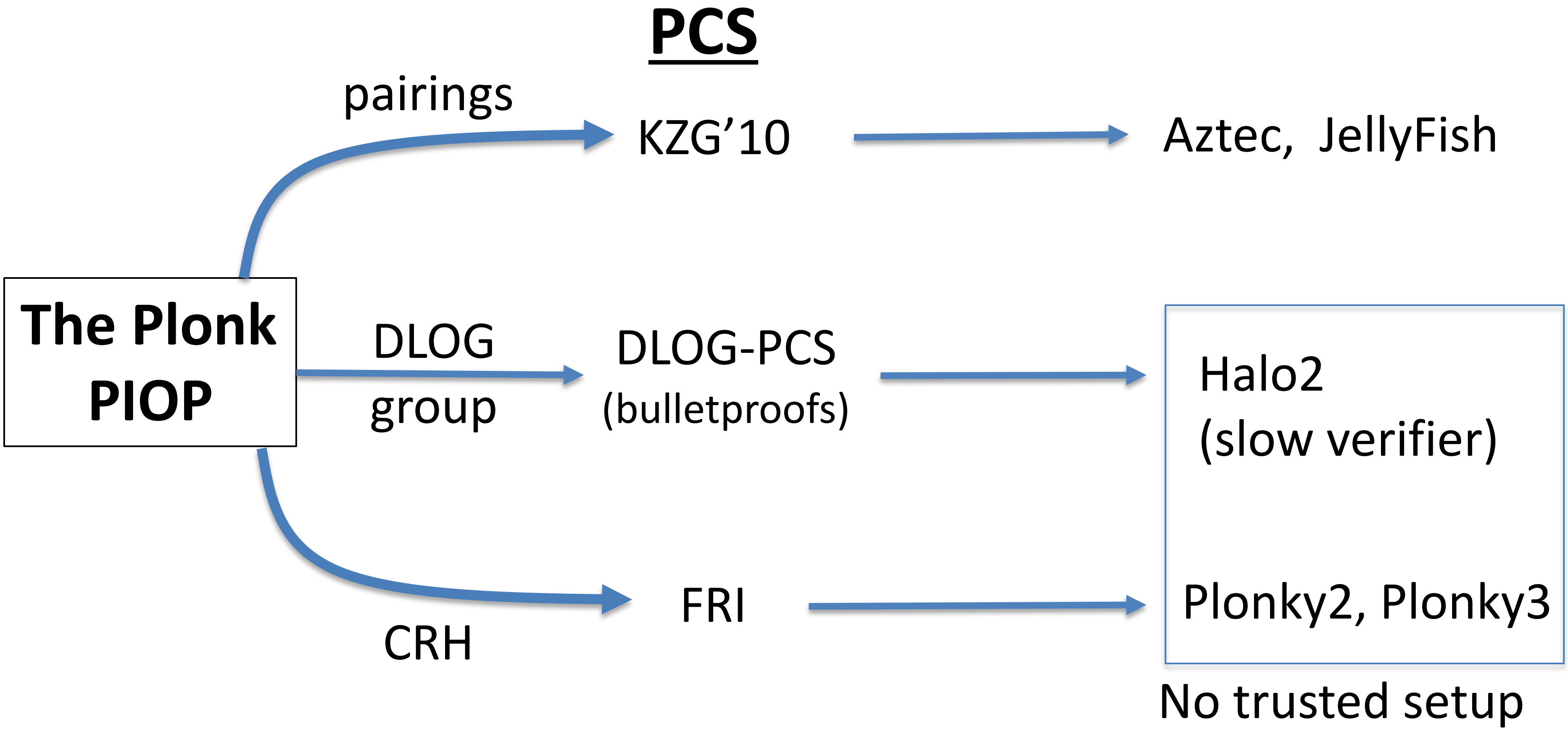
The Plonk poly-IOP (eprint/2019/953)

Gabizon – Williamson – Ciobotaru

Plonk PIOP + Polynomial Commitment $\Rightarrow$ SNARK

(and also a zk-SNARK)

Plonk Systems

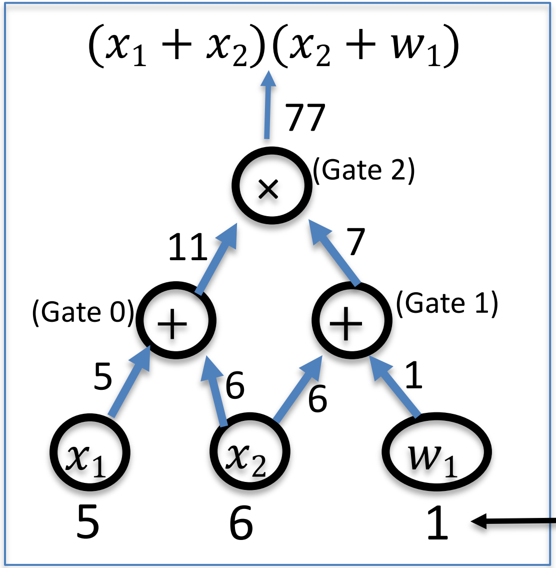


The PLONK PIOP

eprint/2019/953

PLONK: a poly-IOP for a general circuit $C(x, w)$

Step 1: compile circuit to a computation trace (gate fan-in = 2)



example input

The computation trace (arithmetization):

inputs:	5,	6,	1
Gate 0:	5 ,	6 ,	11
Gate 1:	6 ,	1 ,	7
Gate 2:	11,	7,	77

left inputs | right inputs | outputs

Encoding the trace as a polynomial

$|C|$:= total # of gates in C , $|I| := |I_x| + |I_w| =$ # inputs to C

let $d := 3\,|C| + |I|$ (in example, $d = 12$) and $\Omega := \{\, 1, \omega, \omega^2, \ldots, \omega^{d-1} \,\}$

The plan:

prover interpolates a poly. $T \in \mathbb{F}_p^{(\leq d)}[X]$

that encodes the entire trace.

Let's see how …

inputs:	5,	6,	1
Gate 0:	5 ,	6 ,	11
Gate 1:	6 ,	1 ,	7
Gate 2:	11,	7,	77

Encoding the trace as a polynomial

The plan: Prover interpolates $T \in \mathbb{F}_p^{(\leq d)}[X]$ such that

(1) ***T* encodes all inputs**: $T(\omega^{-j}) = \text{input } \#j$ for $j = 1, \ldots, |I|$

(2) ***T* encodes all wires**: $\forall\, l = 0, \ldots, |C| - 1$:

- $T(\omega^{3l})$: left input to gate $\#l$
- $T(\omega^{3l+1})$: right input to gate $\#l$
- $T(\omega^{3l+2})$: output of gate $\#l$

Plonk PIOP:
- send oracle for T
- prove T is valid (gates and wires)

Encoding the trace as a polynomial

In our example, Prover interpolates $T(X)$ such that:

inputs:	$T(\omega^{-1}) = 5,$	$T(\omega^{-2}) = 6,$	$T(\omega^{-3}) = 1,$
gate 0:	$T(\omega^{0}) = 5,$	$T(\omega^{1}) = 6,$	$T(\omega^{2}) = 11,$
gate 1:	$T(\omega^{3}) = 6,$	$T(\omega^{4}) = 1,$	$T(\omega^{5}) = 7,$
gate 2:	$T(\omega^{6}) = 11,$	$T(\omega^{7}) = 7,$	$T(\omega^{8}) = 77$

degree(T) = 11

inputs:	5,	6,	1
Gate 0:	5 ,	6 ,	11
Gate 1:	6 ,	1 ,	7
Gate 2:	11,	7,	77

Prover can use FFT to compute the coefficients of T in time $O(d \log d)$

Step 2: proving validity of T

Prover P($pp, \boldsymbol{x}, \mathbf{w}$) — Verifier V($vp, \boldsymbol{x}$)

build T(X) $\in \mathbb{F}_p^{(\leq d)}$[X] —— T ——→

Prover needs to prove that T is a correct computation trace:

(1) T encodes the correct inputs,
(2) every gate is evaluated correctly,
(3) the wiring is implemented correctly,
(4) the output of last gate is 0

(wiring constraints)

inputs:	5 ,	6,	1
Gate 0:	5 ,	6 ,	11
Gate 1:	6 ,	1 ,	7
Gate 2:	11,	7,	77

How? First, let's build some tools.

Towards the Plonk PIOP

Proving properties of committed univariate polynomials

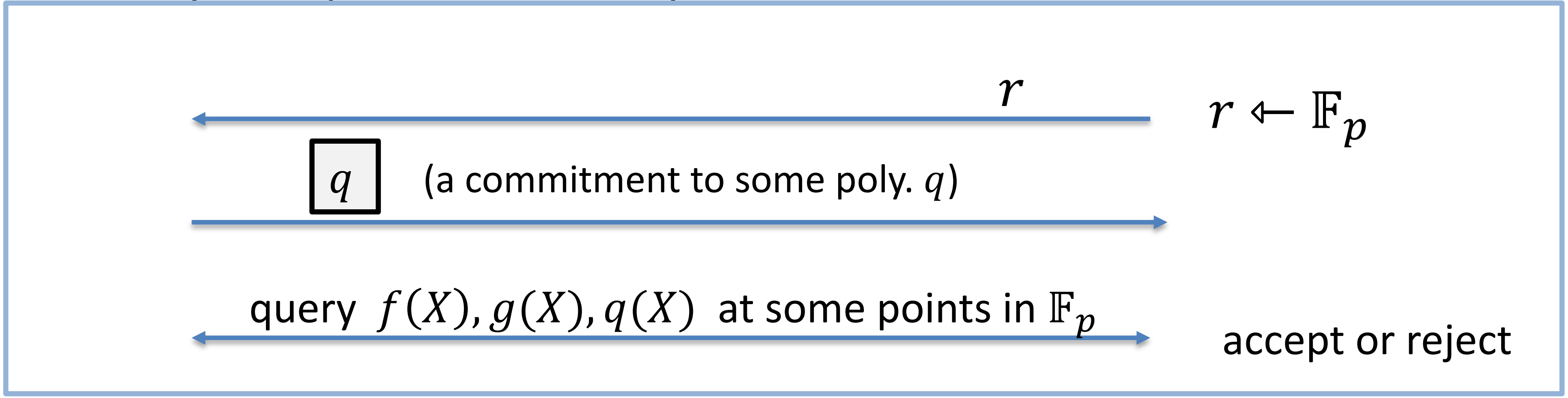
Proving properties of committed polynomials

Prover P(f, g)

Verifier V($\boxed{f}$, $\boxed{g}$)

Goal: convince verifier that $f, g \in \mathbb{F}_p^{(\leq d)}[X]$ satisfy some properties

Proof systems presented as a Poly-IOP:



An example: polynomial equality testing

Prover

$f, g \in \mathbb{F}_p^{(\leq d)}[X]$

Goal: convince verifier that $f = g$

Verifier

f g

query $f(X)$ and $g(X)$ at r

$r \leftarrow \mathbb{F}_p$

learn $f(r), g(r)$

accept if:
$f(r) = g(r)$

Why is this sound?

Why is this sound?

A key fact: for non-zero $f \in \mathbb{F}_p^{(\leq d)}[X]$

$$\text{for } r \leftarrow \mathbb{F}_p : \quad \Pr[\, f(r) = 0 \,] \leq d/p \qquad (*)$$

⇒ suppose $p \approx 2^{256}$ and $d \leq 2^{40}$ then d/p is negligible

⇒ for $r \leftarrow \mathbb{F}_p$: if $f(r) = 0$ then f is identically zero w.h.p

⇒ a simple test if a committed poly. is the zero poly.

SZDL lemma: $(*)$ also holds for **multivariate** polynomials (where d is total degree of f)

Why is this sound?

Suppose $p \approx 2^{256}$ and $d \leq 2^{40}$ so that d/p is negligible

Let $f, g \in \mathbb{F}_p^{(\leq d)}[X]$.

For $r \leftarrow \mathbb{F}_p$, if $f(r) = g(r)$ then $f = g$ w.h.p

$f(r) - g(r) = 0 \quad \Rightarrow \quad f - g = 0$ w.h.p

$\Rightarrow$ a simple equality test for two committed polynomials

The polynomial equality testing protocol

Prover

$f, g \in \mathbb{F}_p^{(\leq d)}[X]$

Goal: convince verifier that $f = g$

Verifier

f g

$r \xleftarrow{\$} \mathbb{F}_p$

query $f(X)$ and $g(X)$ at $X = r$

learn $f(r), g(r)$

accept if:

$f(r) = g(r)$

Lemma: complete and sound assuming d/p is negligible

The compiled proof system

Make non-interactive using Fiat-Shamir

Prover

$f, g \in \mathbb{F}_p^{(\leq d)}[X]$

$y \leftarrow f(r)$

$y' \leftarrow g(r)$

r

$y, \quad \pi_f$ $\quad y', \quad \pi_g$

proof that $y = f(r)$

proof that $y' = g(r)$

Verifier

$com_f \, , com_g$

$r \xleftarrow{\$} \mathbb{F}_p$

learn $f(r), g(r)$

accept if:

(i) $y = y'$ and

(ii) $\pi_f, \ \pi_g$ are valid

Important proof gadgets for univariates

Let Ω be some subset of $\mathbb{F}_p$ of size k.

Let $f \in \mathbb{F}_p^{(\leq d)}[X]$ $\quad (d \geq k)$ $\quad$ Verifier has $\boxed{f}$

Let us construct efficient Poly-IOPs for the following tasks:

Task 1 (**ZeroTest**): prove that f is identically zero on Ω

Task 2 (**SumCheck**): prove that $\sum_{a\in\Omega} f(a) = 0$

Task 3 (**ProdCheck**): prove that $\prod_{a\in\Omega} f(a) = 1$

The vanishing polynomial

Let Ω be some subset of $\mathbb{F}_p$ of size k.

Def: the **vanishing polynomial** of Ω is $Z_\Omega(X) := \prod_{a \in \Omega}(X - a)$

$\deg(Z_\Omega) = k$

Let $\omega \in \mathbb{F}_p$ be a primitive k-th root of unity (so that $\omega^k = 1$).

- if $\Omega = \{1, \omega, \omega^2, \ldots, \omega^{k-1}\} \subseteq \mathbb{F}_p$ then $Z_\Omega(X) = X^k - 1$

$\Rightarrow$ for $r \in \mathbb{F}_p$, evaluating $Z_\Omega(r)$ takes $2 \log_2 k$ field operations

(1) ZeroTest on Ω ($\Omega = \{1, \omega, \omega^2, \ldots, \omega^{k-1}\}$)

Prover P(f)

Verifier V($\boxed{f}$)

$q(X) \leftarrow f(X)/Z_\Omega(X)$

$q \in \mathbb{F}_p^{(\leq d)}[X]$

$r \xleftarrow{\$} \mathbb{F}_p$

verifier evaluates $Z_\Omega(r)$ by itself

query $q(X)$ and $f(X)$ at r

learn $q(r)$, $f(r)$

Lemma: f is zero on Ω if and only if $f(X)$ is divisible by $Z_\Omega(X)$

accept if $f(r) \stackrel{?}{=} q(r) \cdot Z_\Omega(r)$

(implies that $f(X) = q(X) \cdot Z_\Omega(X)$ w.h.p)

Thm: this protocol is complete and sound, assuming d/p is negligible.

(1) ZeroTest on Ω ($\Omega = \{ 1, \omega, \omega^2, \ldots, \omega^{k-1} \}$)

Prover P(f)

Verifier V($\boxed{f}$)

$q(X) \leftarrow f(X)/Z_\Omega(X)$

$q \in \mathbb{F}_p^{(\leq d)}[X]$ →

$r \xleftarrow{\$} \mathbb{F}_p$

verifier evaluates $Z_\Omega(r)$ by itself

query $q(X)$ and $f(X)$ at r ↔

learn $q(r)$, $f(r)$

accept if $f(r) \stackrel{?}{=} q(r) \cdot Z_\Omega(r)$

(implies that $f(X) = q(X) \cdot Z_\Omega(X)$ w.h.p)

Lemma: f is zero on Ω if and only if $f(X)$ is divisible by $Z_\Omega(X)$

Verifier time: O(log k) and two poly queries (but can be batched)

Prover time: dominated by time to compute $q(X)$ [and commit to $q(X)$]

(3) Product check on Ω : $\prod_{a\in\Omega} f(a) = 1$

Set $t \in \mathbb{F}_p^{(\leq k)}[X]$ to be the degree-d polynomial:

$$t(1) = f(1), \qquad t(\omega^s) = \prod_{i=0}^{s} f(\omega^i) \quad \text{for } s = 1, \dots, k-1$$

Then $t(\omega^{k-1}) = \prod_{a\in\Omega} f(a)$

and $t(\omega \cdot x) = t(x) \cdot f(\omega \cdot x)$ for all $x \in \Omega$ (including $x = \omega^{k-1}$)

Lemma: if (1) $t(\omega^{k-1}) = 1$ and

(2) $t_1(x) := t(\omega \cdot x) - t(x) \cdot f(\omega \cdot x) = 0 \quad \forall x \in \Omega$

then $\prod_{a\in\Omega} f(a) = 1$

(3) Product check on Ω (unoptimized)

Prover P(f) | Verifier V($\boxed{f}$)

construct $t(X) \in \mathbb{F}_p^{(\leq k)}$, $t_1(X) := t(\omega \cdot X) - t(X) \cdot f(\omega \cdot X)$

and $q(X) := t_1(X)/(X^k - 1) \in \mathbb{F}_p^{(\leq d)}$

$\boxed{q\,,\quad t}$ →

$r \leftarrow \mathbb{F}_p$

query $t(X)$ at ω^{k-1}, r, ωr →

query $q(X)$ at r , and $f(X)$ at ωr →

learn $t(\omega^{k-1})$, $t(r)$, $t(\omega r)$, $q(r)$, $f(\omega r)$

$t_1(\Omega) = 0 \quad \Leftarrow$

accept if $t(\omega^{k-1}) \stackrel{?}{=} 1$ and

$t(\omega r) - t(r) f(\omega r) \stackrel{?}{=} q(r) \cdot (r^k - 1)$

Complete and sound, assuming $\deg(t_1)/p = (k + d)/p$ is negligible.

Same works for rational functions: $\prod_{a\in\Omega}(f/g)(a) = 1$

Prover P(f, g)

Verifier V(f , g)

Set $t \in \mathbb{F}_p^{(\leq k)}[X]$ to be the degree-k polynomial:

$$t(1) = f(1)/g(1), \qquad t(\omega^s) = \prod_{i=0}^{s} f(\omega^i)/g(\omega^i) \quad \text{for } s = 1, \ldots, k-1$$

Lemma: if (i) $t(\omega^{k-1}) = 1$ and

(ii) $t(\omega \cdot x) \cdot g(\omega \cdot x) = t(x) \cdot f(\omega \cdot x)$ for all $x \in \Omega$

then $\prod_{a\in\Omega} f(a)/g(a) = 1$

(4) Another useful gadget: permutation check

Let f, g polynomials in $\mathbb{F}_p^{(\leq d)}[X]$. Verifier has $\boxed{f}$, $\boxed{g}$.

Prover wants to prove that $\left(f(1), f(\omega), f(\omega^2), \ldots, f(\omega^{k-1}) \right) \in \mathbb{F}_p^k$

is a permutation of $\left(g(1), g(\omega), g(\omega^2), \ldots, g(\omega^{k-1}) \right) \in \mathbb{F}_p^k$

$\Rightarrow$ Proves that $g(\Omega)$ is the same as $f(\Omega)$, just permuted

(4) Another useful gadget: permutation check

Prover P(f, g) | Verifier V($\boxed{f}$, $\boxed{g}$)

Let $\hat{f}(X) = \prod_{a \in \Omega}(X - f(a))$ and $\hat{g}(X) = \prod_{a \in \Omega}(X - g(a))$

Then: $\hat{f}(X) = \hat{g}(X) \iff g(\Omega)$ is a permutation of $f(\Omega)$

$\xleftarrow{\quad r \quad}$ $r \xleftarrow{\$} \mathbb{F}_p$

prove that $\hat{f}(r) = \hat{g}(r)$

prod-check: $\frac{\hat{f}(r)}{\hat{g}(r)} = \prod_{a \in \Omega} \left(\frac{r - f(a)}{r - g(a)} \right) = 1$

$\longleftrightarrow$ implies $\hat{f}(X) = \hat{g}(X)$ w.h.p

accept or reject

[Lipton's trick, 1989]

(4') Permutation check on pairs

Let f_1, f_2, g_1, g_2 be polynomials in $\mathbb{F}_p^{(\leq d)}[X]$.

Prover wants to prove that the k pair

$$\Big(\big(f_1(1), f_2(1)\big), \ldots, \big(f_1(\omega^{k-1}), f_2(\omega^{k-1})\big)\Big) \in (\mathbb{F}_p^2)^k$$

are a permutation of

$$\Big(\big(g_1(1), g_2(1)\big), \ldots, \underbrace{\big(g_1(\omega^{k-1}), g_2(\omega^{k-1})\big)}_{\text{one pair}}\Big) \in (\mathbb{F}_p^2)^k$$

(4') Permutation check on pairs

Define: $\hat{f}(X,Y) \coloneqq \prod_{a\in\Omega} (X - Y \cdot f_1(a) - f_2(a))$ and

$\hat{g}(X,Y) \coloneqq \prod_{a\in\Omega} (X - Y \cdot g_1(a) - g_2(a))$

Lemma: $\hat{f}(X,Y) = \hat{g}(X,Y)$ if and only if

$\big(f_1(a), f_2(a)\big)_{a\in\Omega}$ is a permutation of $\big(g_1(a), g_2(a)\big)_{a\in\Omega}$

To prove, use the fact that $\mathbb{F}_p[X,Y]$ is a unique factorization domain

Now: $\hat{f}(X,Y) = \hat{g}(X,Y)$ can be checked using a product check (using $X, Y \leftarrow \mathbb{F}_p$)

The protocol

Prover P(f_1, f_2, g_1, g_2) Verifier V(f_1, f_2 , g_1, g_2)

r, s

$r, s \leftarrow \mathbb{F}_p$

prove that $\hat{f}(r, s) = \hat{g}(r, s)$:

ProdCheck: $$\prod_{a \in \Omega} \left(\frac{r - s \cdot f_1(a) - f_2(a)}{r - s \cdot g_1(a) - g_2(a)} \right) = 1$$

by Schwartz-Zippel

implies $\hat{f}(X, Y) = \hat{g}(X, Y)$ w.h.p

accept or reject

Complete and sound, assuming $(k + d)/p$ is negligible.

(5) final gadget: prescribed permutation check

$W: \Omega \to \Omega$ is a **permutation of** Ω if $\quad \forall i \in [k]: \; W(\omega^i) = \omega^j \;$ is a bijection

example $(k = 3)$: $\quad W(\omega^0) = \omega^2, \quad W(\omega^1) = \omega^0, \quad W(\omega^2) = \omega^1$

Let f, g polynomials in $\mathbb{F}_p^{(\leq d)}[X]$. Verifier has $\boxed{f}$, $\boxed{g}$, $\boxed{W}$.

<u>Goal</u>: prover wants to prove that $\quad f(y) = g(W(y)) \quad$ for all $\; y \in \Omega$

$\Rightarrow$ Proves that $g(\Omega)$ is the same as $f(\Omega)$, permuted by the prescribed W

Prescribed permutation check

How? Use a zero-test to prove $f(y) - g(W(y)) = 0$ on Ω

The problem: the polynomial $f(y) - g(W(y))$ has degree kd

$\Rightarrow$ prover would need to manipulate polynomials of degree kd

$\Rightarrow$ quadratic time prover !! (goal: linear time prover)

Goal: reduce this to a perm. check on pairs for degree-d poly (not kd)

Prescribed permutation check

Observation:

if $\big(W(a), f(a)\big)_{a\in\Omega}$ is a permutation of $\big(a, g(a)\big)_{a\in\Omega}$

then $f(y) = g(W(y))$ for all $y \in \Omega$

Proof by example: $W(\omega^0) = \omega^2$, $W(\omega^1) = \omega^0$, $W(\omega^2) = \omega^1$

Right tuple: $(\omega^0, g(\omega^0))$, $(\omega^1, g(\omega^1))$, $(\omega^2, g(\omega^2))$

Left tuple: $(\omega^2, f(\omega^0))$, $(\omega^0, f(\omega^1))$, $(\omega^1, f(\omega^2))$

So: permutation check on pairs $\Rightarrow$ prescribed permutation check

Summary of proof gadgets

prescribed permutation check

permutation check on pairs

product check, sum check

zero test on Ω

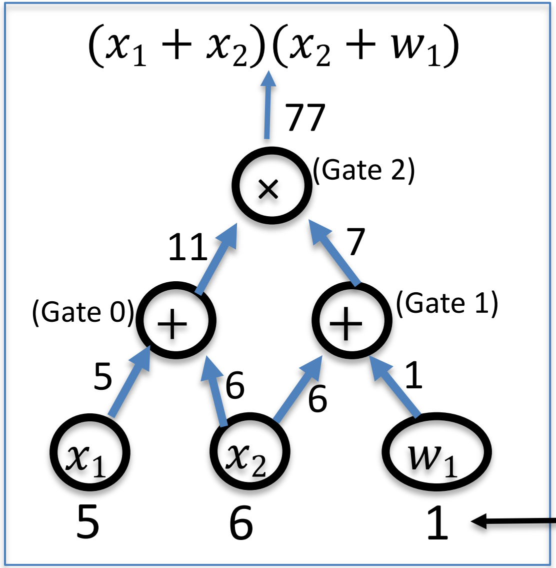
polynomial equality testing

The PLONK Poly-IOP for general circuits

eprint/2019/953

PLONK: a poly-IOP for a general circuit $C(x, w)$

Step 1: compile circuit to a computation trace (gate fan-in = 2)



example input

The computation trace (arithmetization):

inputs:	5,	6,	1
Gate 0:	5 ,	6 ,	11
Gate 1:	6 ,	1 ,	7
Gate 2:	11,	7,	77

left inputs

right inputs

outputs

Encoding the trace as a polynomial

$|C|$:= total # of gates in C , $|I|$:= $|I_x|$ + $|I_w|$ = # inputs to C

let $d := 3\,|C| + |I|$ (in example, $d = 12$) and $\Omega := \{\, 1, \omega, \omega^2, \ldots, \omega^{d-1} \,\}$

The plan:

prover interpolates a poly. $T \in \mathbb{F}_p^{(\leq d)}[X]$

that encodes the entire trace.

Let's see how …

inputs:	5,	6,	1
Gate 0:	5 ,	6 ,	11
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Encoding the trace as a polynomial

The plan: Prover interpolates $T \in \mathbb{F}_p^{(\leq d)}[X]$ such that

(1) ***T* encodes all inputs**: $T(\omega^{-j}) = \text{input } \#j$ for $j = 1, \ldots, |I|$

(2) ***T* encodes all wires**: $\forall\, l = 0, \ldots, |C| - 1$:

- $T(\omega^{3l})$: left input to gate $\#l$
- $T(\omega^{3l+1})$: right input to gate $\#l$
- $T(\omega^{3l+2})$: output of gate $\#l$

inputs:	5,	6,	1
Gate 0:	5 ,	6 ,	11
Gate 1:	6 ,	1 ,	7
Gate 2:	11,	7,	77

Encoding the trace as a polynomial

In our example, Prover interpolates $T(X)$ such that:

inputs:	$T(\omega^{-1}) = 5,$	$T(\omega^{-2}) = 6,$	$T(\omega^{-3}) = 1,$
gate 0:	$T(\omega^{0}) = 5,$	$T(\omega^{1}) = 6,$	$T(\omega^{2}) = 11,$
gate 1:	$T(\omega^{3}) = 6,$	$T(\omega^{4}) = 1,$	$T(\omega^{5}) = 7,$
gate 2:	$T(\omega^{6}) = 11,$	$T(\omega^{7}) = 7,$	$T(\omega^{8}) = 77$

degree(T) = 11

inputs:	5,	6,	1
Gate 0:	5 ,	6 ,	11
Gate 1:	6 ,	1 ,	7
Gate 2:	11,	7,	77

Prover can use FFT to compute the coefficients of T in time $O(d \log d)$

Step 2: proving validity of T

Prover P(S_p, $\boldsymbol{x}$, $\mathbf{w}$) — Verifier V(S_v, $\boldsymbol{x}$)

build $T(X) \in \mathbb{F}_p^{(\le d)}[X]$ → T

Prover needs to prove that T is a correct computation trace:

(1) T encodes the correct inputs,
(2) every gate is evaluated correctly,
(3) the wiring is implemented correctly,
(4) the output of last gate is 0

Proving (4) is easy: prove $T(\omega^{3|C|-1}) = 0$

(wiring constraints)

inputs:	5 ,	6,	1
Gate 0:	5 ,	6 ,	11
Gate 1:	6 ,	1 ,	7
Gate 2:	11,	7,	77

Proving (1): T encodes the correct inputs

Both prover and verifier interpolate a polynomial $v(X) \in \mathbb{F}_p^{(\leq |I_x|)}[X]$ that encodes the x-inputs to the circuit:

$$\text{for } j = 1, \ldots, |I_x|: \qquad v(\omega^{-j}) = \text{input \#j}$$

In our example: $v(\omega^{-1}) = 5, \quad v(\omega^{-2}) = 6$. (v is linear)

constructing $v(X)$ takes time proportional to the size of input x

$\Rightarrow$ verifier has time do this

Proving (1): T encodes the correct inputs

Both prover and verifier interpolate a polynomial $v(X) \in \mathbb{F}_p^{(\leq |I_x|)}[X]$ that encodes the x-inputs to the circuit:

$$\text{for } j = 1, \ldots, |I_x|: \qquad v(\omega^{-j}) = \text{input \#j}$$

Let $\Omega_{\text{inp}} := \{\, \omega^{-1}, \omega^{-2}, \ldots, \omega^{-|I_x|} \,\} \subseteq \Omega$ (points encoding the input)

Prover proves (1) by using a ZeroTest on Ω_{inp} to prove that

$$T(y) - v(y) = 0 \qquad \forall\, y \in \Omega_{\text{inp}}$$

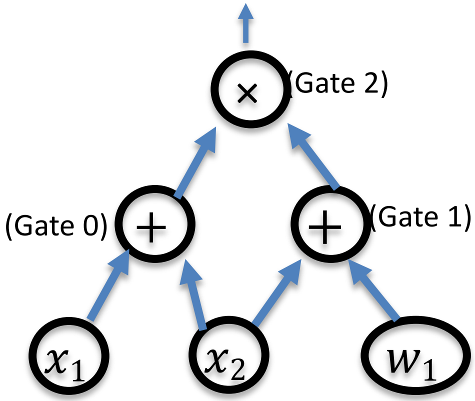
Proving (2): every gate is evaluated correctly

Idea: encode gate types using a *selector* polynomial S(X)

define S(X) ∈ $\mathbb{F}_p^{(\leq d)}$[X] such that $\forall\, l = 0, \ldots, |C| - 1$:

S(ω^{3l}) = 1 if gate #l is an addition gate

S(ω^{3l}) = 0 if gate #l is a multiplication gate



inputs:	5 ,	6,	1	$S(X)$	
Gate 0 (ω^0):	5 ,	6 ,	11	1	(+)
Gate 1 (ω^3):	6 ,	1 ,	7	1	(+)
Gate 2 (ω^6):	11,	7,	77	0	(×)

Proving (2): every gate is evaluated correctly

Idea: encode gate types using a *selector* polynomial S(X)

define $S(X) \in \mathbb{F}_p^{(\leq d)}[X]$ such that $\forall\, l = 0, \ldots, |C| - 1$:

$S(\omega^{3l}) = 1$ if gate $\#l$ is an addition gate

$S(\omega^{3l}) = 0$ if gate $\#l$ is a multiplication gate

Then $\forall\, y \in \Omega_{\text{gates}} := \{\, 1, \omega^3, \omega^6, \omega^9, \ldots, \omega^{3(|C|-1)} \,\}$:

$$S(y)\cdot[T(y) + T(\omega y)] \quad + \quad (1 - S(y))\cdot T(y)\cdot T(\omega y) \quad = \quad T(\omega^2 y)$$

left input — right input — left input — right input — output

Proving (2): every gate is evaluated correctly

Setup(C) → pp:=S and vp:= (S)

Prover P(pp, $\boldsymbol{x}$, $\mathbf{w}$)

Verifier V(vp, $\boldsymbol{x}$)

build T(X) $\in \mathbb{F}_p^{(\le d)}[X]$ —— T ——→

Prover uses ZeroTest to prove that for all $\forall\, y \in \Omega_{gates}$:

$$S(y)\cdot[T(y) + T(\omega y)] + (1 - S(y))\cdot T(y)\cdot T(\omega y) - T(\omega^2 y) = 0$$

Proving (3): T respects the wires of C

Copy constraints:

$$\begin{cases} T(\omega^{-2}) = T(\omega^{1}) = T(\omega^{3}) \\ T(\omega^{-1}) = T(\omega^{0}) \\ T(\omega^{2}) = T(\omega^{6}) \\ T(\omega^{-3}) = T(\omega^{4}) \end{cases}$$

<u>example:</u> $x_1=5,\ x_2=6\ ,\ w_1=1$

	$\omega^{-1},\ \omega^{-2},\ \omega^{-3}$:	**5,**	**6,**	**1**
0:	$\omega^{0},\ \omega^{1},\ \omega^{2}$:	**5,**	**6,**	**11**
1:	$\omega^{3},\ \omega^{4},\ \omega^{5}$:	**6,**	**1,**	**7**
2:	$\omega^{6},\ \omega^{7},\ \omega^{8}$:	**11,**	**7,**	**77**

Define a polynomial W: $\Omega \to \Omega$ that implements a rotation:

$W(\omega^{-2}, \omega^{1}, \omega^{3}) = (\omega^{1}, \omega^{3}, \omega^{-2})$, $W(\omega^{-1}, \omega^{0}) = (\omega^{0}, \omega^{-1})$, ...

<u>Lemma</u>: $\forall\, y \in \Omega$: $T(y) = T(W(y))$ $\Rightarrow$ wire constraints are satisfied

Proving (3): T respects the wires of C

Copy constraints:

$T(\omega^{-2}) = T(\omega^{1}) = T(\omega^{3})$

$T(\omega^{-1}) = T(\omega^{0})$

$T(\omega^{2}) = T(\omega^{6})$

example: $x_1=5,\ x_2=6,\ w_1=1$

	$\omega^{-1}, \omega^{-2}, \omega^{-3}$:	**5,**	**6,**	**1**
0:	$\omega^{0}, \omega^{1}, \omega^{2}$:	**5,**	**6,**	**11**
1:	$\omega^{3}, \omega^{4}, \omega^{5}$:	**6,**	**1,**	**7**
				77

Proved using a prescribed permutation check

Define a polynom ... $\to \Omega$ that implements a rotation:

$W(\omega^{-2}, \omega^{1}, \omega^{3})$ ... $\omega^{3}, \omega^{-2})$, $W(\omega^{-1}, \omega^{0}) = (\omega^{0}, \omega^{-1})$, ...

Lemma: $\forall\, y \in \Omega$: $T(y) = T(W(y)) \Rightarrow$ wire constraints are satisfied

The complete Plonk Poly-IOP (and SNARK)

Setup(C) → $pp := (S, W)$ and $vp := (\boxed{S}$ and $\boxed{W})$ (untrusted)

<u>Prover P($pp, \boldsymbol{x}, \mathbf{w}$)</u>

build $T(X) \in \mathbb{F}_p^{(\leq d)}[X]$

$\boxed{T}$ →

<u>Verifier V($vp, \boldsymbol{x}$)</u>

build $v(X) \in \mathbb{F}_p^{(\leq |I_x|)}[X]$

Prover proves:

gates: (1) $S(y) \cdot [T(y) + T(\omega y)] + (1 - S(y)) \cdot T(y) \cdot T(\omega y) - T(\omega^2 y) = 0$; $\forall\, y \in \Omega_{gates}$

inputs: (2) $T(y) - v(y) = 0$ $\forall\, y \in \Omega_{inp}$

wires: (3) $T(y) - T(W(y)) = 0$ (using prescribed perm. check) $\forall\, y \in \Omega$

output: (4) $T(\omega^{3|C|-1}) = 0$ (output of last gate = 0)

The complete Plonk Poly-IOP (and SNARK)

Setup(C) → $pp := (S, W)$ and $vp := (\boxed{S}$ and $\boxed{W})$ (untrusted)

<u>Prover P($pp, \boldsymbol{x}, \mathbf{w}$)</u>

build $T(X) \in \mathbb{F}_p^{(\leq d)}[X]$

$\boxed{T}$ →

<u>Verifier V($vp, \boldsymbol{x}$)</u>

build $v(X) \in \mathbb{F}_p^{(\leq |I_x|)}[X]$

<u>Thm</u>: The Plonk Poly-IOP is complete and knowledge sound, assuming $7|C|/p$ is negligible

(eprint/2019/953)

Many extensions …

Plonk proof: a short proof (O(1) commitments), fast verifier

The SNARK can be made into a zk-SNARK

Main challenge: reduce prover time

- **Hyperplonk**: replace Ω with $\{0,1\}^t$ (where $t = \log_2|\Omega|$)
  - The polynomial T is now a multilinear polynomial in t variables
  - ZeroTest is replaced by a multilinear SumCheck (linear time)

A generalization: plonkish arithmetization

Plonk for circuits with gates other than $+$ and $\times$ on rows:

Plonkish computation trace: (also used in AIR)

u1	v1	w1	t1	r1
u2	v2	w2	t2	r2
u3	v3	w3	t3	r3
u4	v4	w4	t4	r4
u5	v5	w5	t5	r5
u6	v6	w6	t6	r6
u7	v7	w7	t7	r7
u8	v8	w8	t8	r8

output

An example custom gate:

$$\forall\, y \in \Omega: \quad v(y\omega) + w(y) \cdot t(y) - t(y\omega) = 0$$

All such gate checks are included in the gate check

A generalization: plonkish arithmetization

Plonk for circuits with gates other than + and × on rows:

Plonkish computation trace: (also used in AIR)

					S(X)
u1	v1	w1	t1	r1	0
u2	v2	w2	t2	r2	0
u3	v3	w3	t3	r3	1
u4	v4	w4	t4	r4	0
u5	v5	w5	t5	r5	1
u6	v6	w6	t6	r6	0
u7	v7	w7	t7	r7	0
u8	v8	w8	t8	r8	1

output

An example custom gate:

$$\forall\, y \in \Omega:\ \ S(X) \cdot [v(y\omega) + w(y) \cdot t(y) - t(y\omega)] = 0$$

Selector poly $S(X)$ can choose when to apply gate

THE END