CS251 Fall 2020

(cs251.stanford.edu)

Building a SNARK

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Recap: high-level goals

- Private transactions on a public blockchain
- Blockchain scaling, such as proof-based Rollup
- Privately prove compliance, such as a private proof of solvency

Recap: non-interactive proof systems (for NP)

Public arithmetic circuit: $C(\boldsymbol{x}, \boldsymbol{w}) \rightarrow \mathbb{F}_p$

public statement in $\mathbb{F}_p^n$ $\boldsymbol{x}$

$\boldsymbol{w}$ secret witness in $\mathbb{F}_p^m$

Let $\boldsymbol{x} \in \mathbb{F}_p^n$. Two standard goals for prover P:

(1) **<u>Soundness</u>**: convince Verifier that $\exists \boldsymbol{w}$ s.t. $C(\boldsymbol{x}, \boldsymbol{w}) = 0$

(e.g., $\exists \boldsymbol{w}$ such that [$H(\boldsymbol{w}) = \boldsymbol{x}$ and $0 < \boldsymbol{w} < 2^{60}$])

(2) **<u>Knowledge</u>**: convince Verifier that P "knows" $\boldsymbol{w}$ s.t. $C(\boldsymbol{x}, \boldsymbol{w}) = 0$

(e.g., P knows a $\boldsymbol{w}$ such that $H(\boldsymbol{w}) = \boldsymbol{x}$)

Non-interactive Proof Systems (for NP)

A **non-interactive proof system** is a triple (S, P, V):

- **S**(C) $\rightarrow$ public parameters (S_p, S_v) for prover and verifier

  (S_p, S_v) is called a *reference string*

- **P**(S_p, $\boldsymbol{x}$, $\boldsymbol{w}$) $\rightarrow$ proof π

- **V**(S_v, $\boldsymbol{x}$, $\boldsymbol{\pi}$) $\rightarrow$ accept or reject

proof systems: properties (informal)

Prover P(***pp***, x, w) — Verifier V (***pp***, x, π)

proof π →

accept or reject

Complete: $\forall x, w$: $C(x, w) = 0 \Rightarrow$ V(S_v, x, P(S_p, x, w)) = accept

Proof of knowledge: V accepts $\Rightarrow$ P "knows" w s.t. $C(x, w) = 0$

in some cases, **soundness** is sufficient: $\exists w$ s.t. $C(x, w) = 0$

Zero knowledge (optional): (x, π) "reveals nothing" about w

SNARK: succinct argument of knowledge

Goal: P wants to show that it knows $\boldsymbol{w}$ s.t. $C(\boldsymbol{x}, \boldsymbol{w}) = 0$

Succinct:

- Proof π should be **short** [i.e., $|\pi| = O(\mathbf{log}(|\boldsymbol{C}|), \lambda)$]
- Verifying π should be **fast** [i.e., time(V) = $O(|x|, \mathbf{log}(|\boldsymbol{C}|), \lambda)$]

note: if SNARK is zero-knowledge, then called a **zkSNARK**

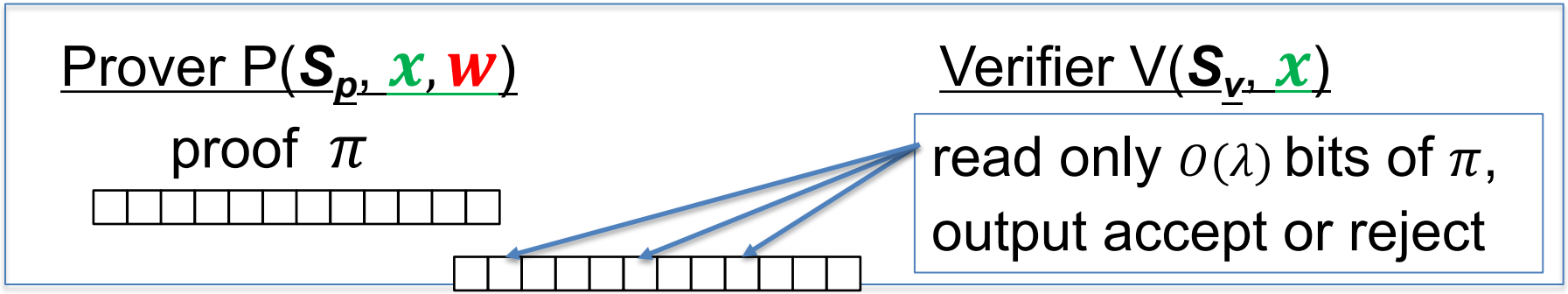
A simple PCP-based SNARK

[Kilian’92, Micali’94]

A simple construction: PCP-based SNARK

The PCP theorem: Let $C(x, w)$ be an arithmetic circuit.

there is a proof system that for every x proves $\exists w: C(x, w) = 0$ as follows:

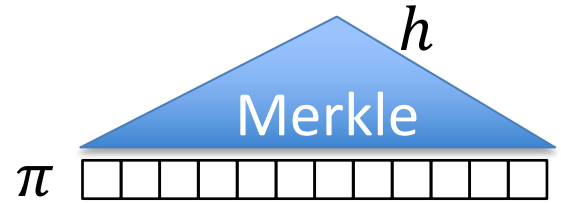


V always accepts valid proof. If no w, then V rejects with high prob.

size of proof is $poly(|C|)$. (not succinct)

Converting a PCP proof to a SNARK

Prover P(S_p, x, w) Verifier V(S_v, x)



Merkle root h → 1 hash

open k positions of π ($k = O(\lambda)$) →

$O(k \log |C|)$ hashes ←

k opening and Merkle proofs →

output accept or reject

Verifier sees $O(\lambda \log |C|)$ data ⇒ succinct proof.

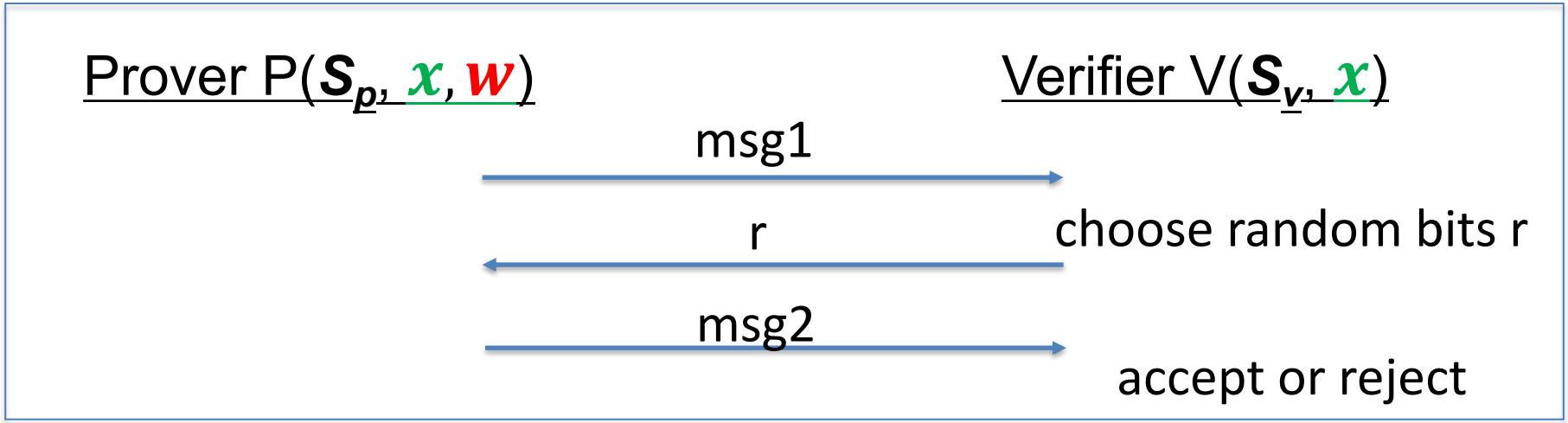
Problem: **interactive**

Making the proof non-interactive

The **Fiat-Shamir heuristic:**

- public-coin interactive protocol ⇒ non-interactive protocol

  public coin: all verifier randomness is public (no secrets)



Making the proof non-interactive

Fiat-Shamir heuristic: $H: M \to R$ a cryptographic hash function

- idea: prover generates random bits on its own (!)

Prover P(S_p, x, w)

generate msg1

r ⇐ H(x, msg1)

generate msg2

π = (msg1, msg2)

$|\pi| = O(\lambda \log |C|)$

Verifier V(S_v, x)

r ⇐ H(x, msg1)

accept or reject

Thm: this is a secure SNARK assuming H is a random oracle

Why is this an argument of knowledge? (can skip)

Let's build an extractor E for the interactive protocol:

- After prover commits to Merkle root of proof

  E asks prover to open many batches of $k = O(\lambda)$ positions of π (by rewinding prover)

- E fails to extract cell #j of π if

  (1) prover produces a false Merkle proofs (efficient prover cannot), or

  (2) prover fails (i.e., verifier rejects) whenever j is in batch to open:

  $$\Pr[\text{prover fails}] \geq \Pr[\text{ j in batch }] = 1 - (1 - 1/|\pi|)^k .$$

  so: this cannot happen if k is sufficiently large

$\Rightarrow$ E extracts entire proof π. Once π is known, E can obtain $\boldsymbol{w}$ from π.

Are we done?

Simple transparent SNARK from the PCP theorem

- Use Fiat-Shamir heuristic to make non-interactive
- We will apply Fiat-Shamir in many other settings

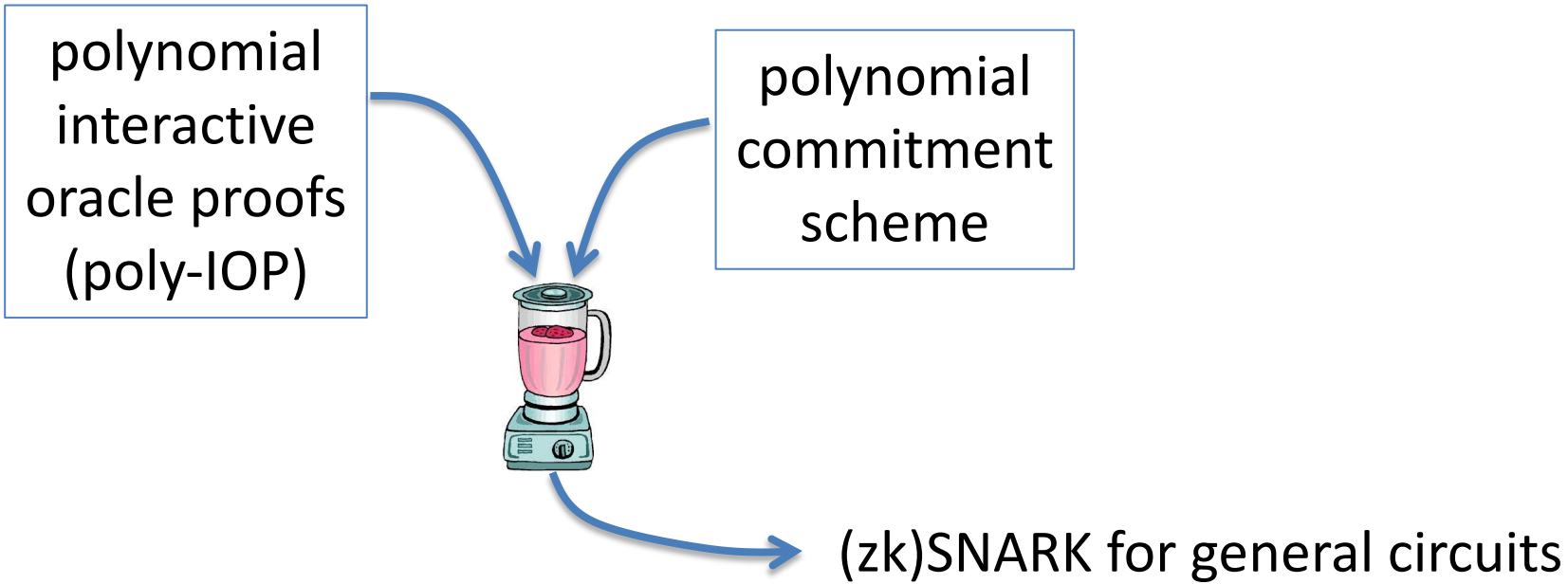
The bad news: an impractical SNARK --- Prover time too high

Better SNARKs: Goal: Time(Prover) = $\tilde{O}(|C|)$

Building an efficient SNARK

General paradigm

Many SNARKs are built in two steps:



Recall: commitments

Two algorithms:

- *commit*(m, r) ⇾ ***com***
- *verify*(m, ***com***, r) ⇾ accept or reject

Properties:

- binding: cannot produce two valid openings for ***com***.
- hiding: ***com*** reveals nothing about committed data

(1) Polynomial commitment schemes

Notation:

Fix a finite field: $\mathbb{F}_p = \{0,1, \ldots, p-1\}$

$\mathbb{F}_p^{(\leq d)}[X]$: all polynomials in $\mathbb{F}_p[\mathrm{X}]$ of degree ≤ d.

(1) Polynomial commitment schemes

- *setup*$(d) \rightarrow pp$, public parameters for polynomials of degree $\leq d$
- *commit*$(pp, f, r) \rightarrow \boldsymbol{com_f}$ commitment to $f \in \mathbb{F}_p^{(\leq d)}[X]$
- *eval*: goal: for a given $\boldsymbol{com_f}$ and $x, y \in \mathbb{F}_p$, prove that $f(x) = y$.

Formally: *eval* = (P, V) is a SNARK for:

statement $st = (pp, \boldsymbol{com_f}, x, y)$ with witness = $w = (f, r)$

where $C(st, w) = 0$ iff

$[\, f(x) = y \text{ and } f \in \mathbb{F}_p^{(\leq d)}[X] \text{ and } commit(pp, f, r) = \boldsymbol{com}_f \,]$

(1) Polynomial commitment schemes

Properties:

- Binding: cannot produce two valid openings (f_1, r_1), (f_2, r_2) for $\boldsymbol{com_f}$.
- eval is an argument of knowledge (can extract (f, r) from a successful prover)
- optional:
  - commitment is hiding
  - eval is zero knowledge

Constructing polynomial commitments

Not today … (see readings or CS355)

simple construction without this requirement

Properties of the best ones:

- transparent setup: no secret randomness in setup
- ***com***$_f$ is constant size (a single group element)
- eval proof size for f $\in \mathbb{F}_p^{(\leq d)}[X]$ is O(log d) group elements
- eval verify time is O(log d) Prover time: O(d)

(2) Polynomial IOP

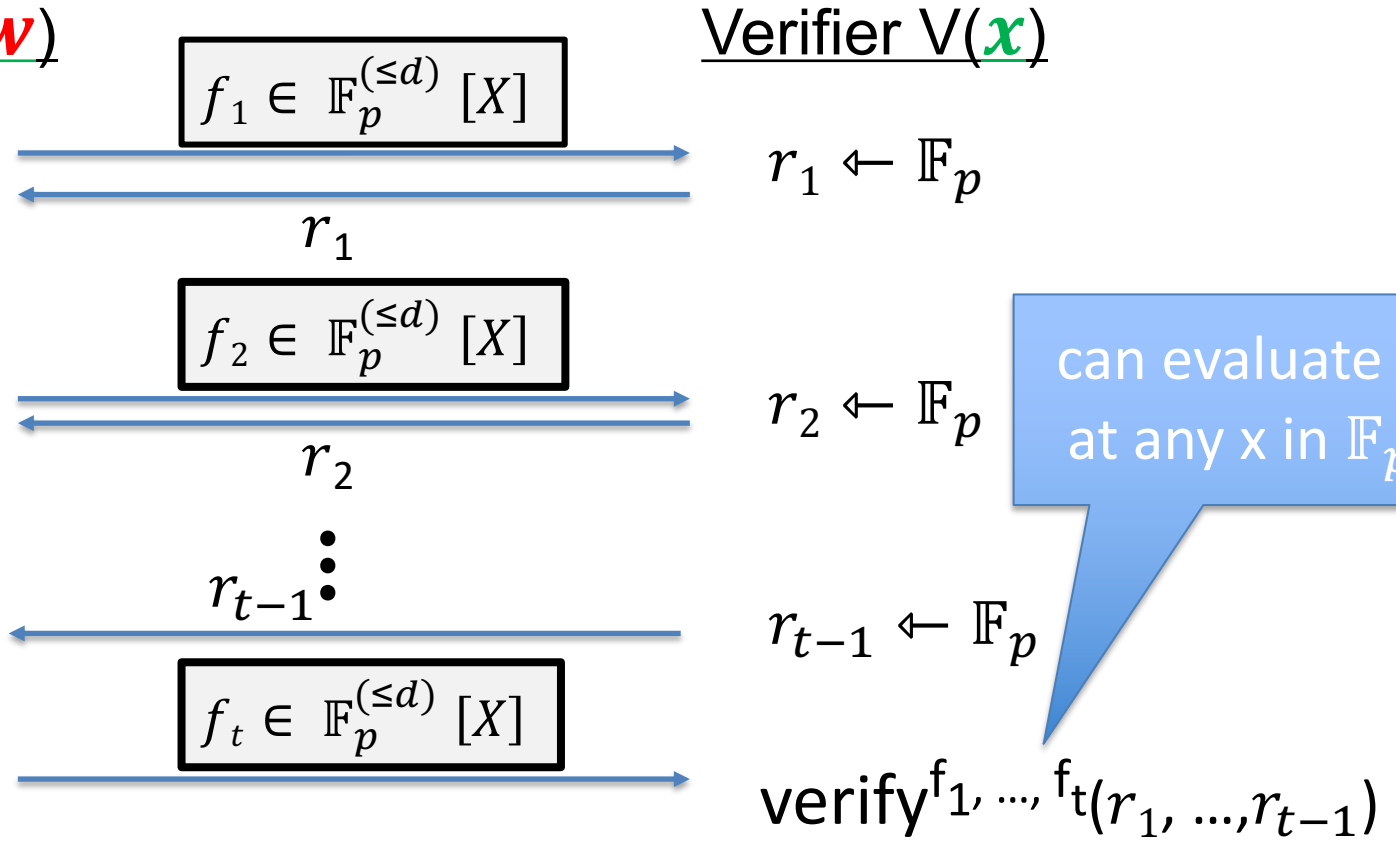
Goal: polynomial commitment scheme ⇒
SNARK for a general circuit $C(x, w)$.

… done using a polynomial-IOP

Fix an arithmetic circuit $C(x, w)$. Let $x \in \mathbb{F}_p^n$.

Poly-IOP: a proof system that proves $\exists w: C(x, w) = 0$ as follows:

(2) Polynomial IOP



Properties

- complete: if $\exists w: C(x, w) = 0$ then verifier always accepts

- Soundness or proof of knowledge: (informal) Let $x \in \mathbb{F}_p^n$.

  P*: a prover that convinces the verifier with prob. ≥ $1/10^6$

  then there is an efficient extractor E s.t.

  $$\Pr\left[\ E(x, f_1, r_1, \ldots, r_{t-1}, f_t) = w \ \text{ s.t. } \ C(x, w) = 0\ \right] \geq 1/10^6$$

- Optional: zero knowledge

The resulting SNARK

Poly-IOP params: #polynomials = t, # eval queries in verify = q

The SNARK:

- During interactive phase of poly-IOP: send t poly commitments
- During poly-IOP verify: run poly-commit eval protocol q times
- Use Fiat-Shamir to make the proof system non-interactive

Length of SNARK proof: t poly-commits + q eval proofs

SNARK verify time: q poly eval proof verifications + time(IOP-verify)

SNARK prover time: t poly commits + time(IOP-prover)

Constructing a Poly-IOP

First some useful tricks …

The fundamental theorem of algebra: for $0 \neq f \in \mathbb{F}_p^{(\leq d)}[X]$

for $r \leftarrow \mathbb{F}_p$: $\Pr[\, f(r) = 0 \,] \leq d/p$

$\Rightarrow$ suppose $p \approx 2^{256}$ and $d \leq 2^{40}$ then d/p is negligible

$\Rightarrow$ for $r \leftarrow \mathbb{F}_p$, if $f(r) = 0$ then f is identically zero w.h.p

$\Rightarrow$ simple zero test for a committed polynomial

Some useful gadgets

Let $\omega \in \mathbb{F}_p$ be a primitive k-th root of unity $(\omega^k = 1)$

Set H := { 1, ω, ω^2, …, ω^{k-1} }.

Let $f \in \mathbb{F}_p^{(\leq d)}[X]$ and $b, c \in \mathbb{F}_p$. $(d \geq k)$

Want poly-IOPs for the following tasks:

Task 1 (**zero-test**): prove that f is identically zero on H

Tast 2 (**sum-check**): prove that $\sum_{a \in H} f(a) = b$

Task 3 (**prod-check**): prove that $\prod_{a \in \mathrm{H}} f(a) = c$

Zero test on H (H = { 1, ω, ω^2, …, ω^{k-1} })

Prover P(f, $\perp$)

Verifier V(f)

$q(X) \leftarrow f(X)/(X^k - 1)$

$q \in \mathbb{F}_p^{(\leq d)}[X]$

$r \leftarrow \mathbb{F}_p$

eval $q(X)$ and $f(X)$ at r

learn $q(r)$, $f(r)$

if f is zero on H then $f(X)$ is divisible by $X^k - 1$

accept if $f(r) \stackrel{?}{=} q(r) \cdot (r^k - 1)$

(implies that $f(X) = q(X)(X^k - 1)$)

Thm: this protocol is complete and sound, assuming d/p is negligible.

Verifier time: O(log k) and two eval verify (but can be done in one)

Product check on H: $\prod_{a\in H} f(a) = 1$

Let $t \in \mathbb{F}_p^{(\le k)}[X]$ be the degree-d polynomial:

$t(1) = f(1), \qquad t(\omega^s) = \prod_{i=0}^{s} f(\omega^i)$ for $s = 1, \ldots, k-1$

Then $t(\omega^{k-1}) = \prod_{a\in H} f(a) = 1$

and $t(\omega \cdot x) = t(x) \cdot f(\omega \cdot x)$ for all $x \in H$ (including $x = \omega^{k-1}$)

Lemma: if (1) $t(\omega^{k-1}) = 1$ and

(2) $t(\omega \cdot x) - t(x) \cdot f(\omega \cdot x) = 0$ for all $x \in H$

then $\prod_{a\in H} f(a) = 1$

Product check on H (unoptimized)

Prover P$((f, c), \perp)$ Verifier V(f)

construct $t(X) \in \mathbb{F}_p^{(\leq k)}$, $t_1(X) = t(\omega \cdot X) - t(X) \cdot f(\omega \cdot X)$

and $q(X) = t_1(X)/(X^k - 1) \in \mathbb{F}_p^{(\leq k)}$

$q,\ t \in \mathbb{F}_p^{(\leq k)}[X]$

$r \leftarrow \mathbb{F}_p$

eval $t(X)$ at ω^{k-1}, r, ωr

learn $t(\omega^{k-1})$, $t(r)$, $t(\omega r)$, $q(r)$, $f(\omega r)$

eval $q(X)$ at r, and $f(X)$ at ωr

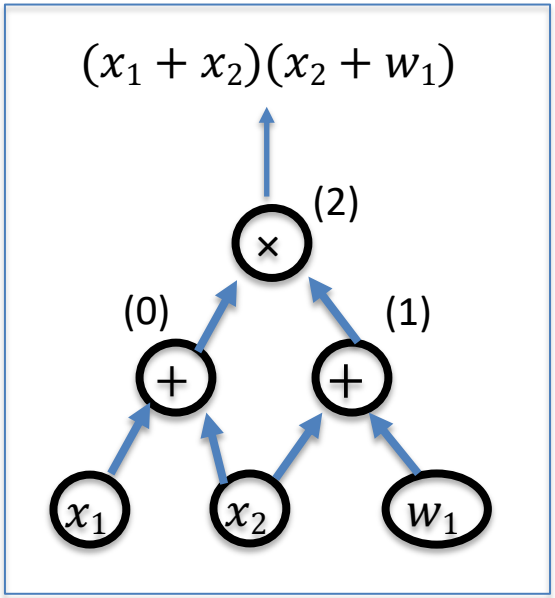
$t_1(H) = 0$:

accept if $t(\omega^{k-1}) \stackrel{?}{=} 1$ and

$t(\omega r) - t(r) f(\omega r) \stackrel{?}{=} q(r) \cdot (r^k - 1)$

PLONK: a poly-IOP for a general circuit $C(x, w)$

Step 1: compile circuit to a sequence of ops (gate fan-in = 2)



compile

Program P

0:	inp_1 , inp_2	:	+
1:	inp_2 , inp_3	:	+
2:	out_0, out_1	:	×

output

(topological sort of gates)

Encoding the inputs to the circuit

Step 2: let $d = 3\,|C| + |I|$ and H = { 1, ω, ω^2, …, ω^{d-1} }

$|C|$ = total # of gates in C , $|I| = |I_x| + |I_w|$ = # inputs to C

- encode the x-inputs to the circuit in a polynomial $v \in \mathbb{F}_p^{(\leq |I_x|)}[X]$

  for $j = 1, .., |I_x|$: $v(\omega^{-j}) =$ input #j

- constructing $v(X)$ takes time proportional to the size of the input

- Let $H_{inp} = \{ \omega^{-1}, \omega^{-2}, \ldots, \omega^{-|I_x|} \}$ (points encoding the input)

Encoding the circuit internal values

The plan: (prover uses FFT to compute coefficients of P in time $d \log_2 d$)

Define a polynomial $P \in \mathbb{F}_p^{(\leq d)}[X]$ such that $\forall\, l = 0, \ldots, |C| - 1$:

- P(ω^{3l}): left input to gate #l
- P(ω^{3l+1}): right input to gate #l
- P(ω^{3l+2}): output of gate #l

and $P(\omega^{-j}) =$ input #j for j = 1, …, $|I|$ (all inputs)

example: x_1=5, x_2=6, w_1=1

	$\omega^{-1}, \omega^{-2}, \omega^{-3}$: 5, 6, 1
0:	$\omega^0, \omega^1, \omega^2$: 5, 6, 11
1:	$\omega^3, \omega^4, \omega^5$: 6, 1, 7
2:	$\omega^6, \omega^7, \omega^8$: 11, 7, 77

Encoding the gates of the circuit

Step 3: encode gate types using a *selector* polynomial S(X)

define S(X) ∈ $\mathbb{F}_p^{(\leq d)}[X]$ such that $\forall\, l = 0, \ldots, |C| - 1$:

S(ω^{3l}) = 1 if gate #l is an addition gate

S(ω^{3l}) = 0 if gate #l is a multiplication gate

Then, $\forall$ x ∈ H_{gates} = { 1, ω^3, ω^6, ω^9, …, $\omega^{3(|C|-1)}$ }:

$$S(x)\cdot[P(x) + P(\omega x)] + (1 - S(x))\cdot P(x)\cdot P(\omega x) = P(\omega^2 x)$$

Encoding the circuit wiring

Step 4: encode the wires of C:

$$\begin{cases} P(\omega^{-2}) = P(\omega^{1}) = P(\omega^{3}) \\ P(\omega^{-1}) = P(\omega^{0}) \\ P(\omega^{2}) = P(\omega^{6}) \\ P(\omega^{-3}) = P(\omega^{4}) \end{cases}$$

example: $x_1=5, x_2=6, w_1=1$

$\omega^{-1}, \omega^{-2}, \omega^{-3}$: **5, 6, 1**

$\omega^{0}, \omega^{1}, \omega^{2}$: **5, 6, 11**

$\omega^{3}, \omega^{4}, \omega^{5}$: **6, 1, 7**

$\omega^{6}, \omega^{7}, \omega^{8}$: **11, 7, 77**

Define a polynomial $W: H \to H$ that implements a rotation:

$$W(\omega^{-2}, \omega^{1}, \omega^{3}) = (\omega^{1}, \omega^{3}, \omega^{-2}) \ , \quad W(\omega^{-1}, \omega^{0}) = (\omega^{0}, \omega^{-1}) \ , \ \ldots$$

Lemma: $\forall\, x \in H$: $P(x) = P(W(x)) \Rightarrow$ wire constraints are satisfied

Encoding the circuit wiring

Problem: the constraint $P(x) = P(W(x))$ has degree d^2

$\Rightarrow$ prover would need to manipulate polynomials of degree d^2

$\Rightarrow$ quadratic time prover !! (goal: linear time prover)

Cute trick: use prod-check proof to reduce this to a constraint of linear degree

Reducing wiring check to a linear degree

Lemma: P(x) = P(W(x)) for all $x \in H$ if and only if $L(Y,Z) \equiv 1$,

where $L(Y,Z) = \prod_{x \in H} \frac{P(x)+Y \cdot W(x)+Z}{P(x)+Y \cdot x+Z}$

To prove that $L(Y,Z) \equiv 1$ do:

(1) verifier chooses random $y, z \in \mathbb{F}_p$

(2) prover builds L_1(X) s.t. $L_1(x) = \frac{P(x)+y \cdot W(x)+z}{P(x)+y \cdot x+z}$ for all $x \in H$

(3) run prod-check to prove $\prod_{x \in H} L_1(x) = 1$

(4) validate L_1: run zero-test to prove $L_2(x) = 0$ for all $x \in H$ where

$$L_2(\mathrm{x}) = (P(x) + y \cdot x + z)\, L_1(x) - (P(x) + y \cdot W(x) + z)$$

The final (S, P, V) SNARK

Setup(C): S_v = (poly commitment to S(X) and W(X))

Prover P($\boldsymbol{x}, \mathbf{w}$) — **Verifier V($S_v, \boldsymbol{x}$)**

build $P(X) \in \mathbb{F}_p^{(\leq d)}[X]$ — P → — build $v(X) \in \mathbb{F}_p^{(\leq |I_x|)}[X]$

Prove:

gates: (1) $S(x)\cdot[P(x) + P(\omega x)] + (1 - S(x))\cdot P(x)\cdot P(\omega x) - P(\omega^2 x) = 0$ $\quad \forall\, x \in H_{gates}$

inputs: (2) $P(x) - v(x) = 0$ $\quad \forall\, x \in H_{inp}$

wires: (3) $P(x) - P(W(x)) = 0$ $\quad \forall\, x \in H$

output: (4) $P(\omega^{3|C|-1}) = 0$ (output of last gate = 0)

Many extensions …

- Can handle circuits with more general gates than + and ×
  - PLOOKUP: efficient SNARK for circuits with lookup tables
- The SNARK can easily be made into a zkSNARK
- Main challenge: reduce prover time

END OF LECTURE

Next lecture: recursive SNARKs