The HHL algorithm

The linear system problem:

Given $A \in \mathbb{C}^{N\times N}$, $\det(A)=1$, $\vec{b} \in \mathbb{C}^N$

output $\tilde{x} \in \mathbb{C}^N$ s.t. $\|\tilde{x} - A^{-1}\vec{b}\|_2 \leq \varepsilon$.

Classically: (1) compute A^{-1} : time $= \tilde{O}(N^3 \cdots)$

(2) better: conjugate Gradient alg:

$$\boxed{\text{time} = O(N \cdot s \cdot K \cdot \log(1/\varepsilon))}$$

s = sparsity. A is s-sparse if # non-zero cells per row $\leq s$.

K = conditionNumber$(A) = |\lambda_{max}| / |\lambda_{min}|$ ← large when matrix is close to non-invertible

$\Rightarrow$ linear in dim N!

Quantum: change the problem (QLSP)

input: $A \in \mathbb{C}^{N\times N}$ Hermitian, $\det(A)=1$, s-sparse.

state $|\vec{b}\rangle = \frac{1}{\|\vec{b}\|} \sum_{j=1}^{N} b_j \cdot |j\rangle \in \mathbb{C}^N$ (*)

output: state $|\tilde{x}\rangle = \sum_{j=1}^{N} x_j |j\rangle \in \mathbb{C}^N$ s.t. $\| |\tilde{x}\rangle - |A^{-1}\vec{b}\rangle \|_2 \leq \varepsilon$

Quantum time (HHL'09, Amb'10):

$$\boxed{O\left(\log N \cdot \zeta^2 \cdot k \cdot \frac{1}{\varepsilon}\right)}$$

log dim

linear in error

20-bit accuracy $\Rightarrow 2^{20}$ time.

(improved in 2017)

Notes:

(1) need to "load" target vector $\bar{b} \in \mathbb{C}^N$

state $(\ast)$ is called a QRAM

easy if $i \to b_i$ is eff. computable classically.

(2) A Hermitian is not a limitation:

if $A \in \mathbb{C}^{N \times N}$ not Hermitian, define $\tilde{A} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}$

if $A\bar{x} = b$ then $\tilde{A} \cdot \begin{pmatrix} 0 \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \bar{b} \\ 0 \end{pmatrix}$.

(3) output $|\tilde{x}\rangle = \sum_{j=1}^{N} x_j |j\rangle \in \mathbb{C}^N$ is a state,

<u>not</u> a cleartext vector.

can measure entries in $\tilde{x}$ or mean of $\tilde{x}$.

or use state $|\tilde{x}\rangle$ in a larger computation.

Tools:

(1) Hamiltonian simulation:

Def: Hermitian matrix $H \in \mathbb{C}^{N \times N}$ can be efficiently simulated if $\forall\, t > 0, \varepsilon > 0$, $\exists$ quantum circuit U_t with $\mathrm{poly}(\log N, t, 1/\varepsilon)$ gates s.t. $\| U_t - e^{-iHt} \| < \varepsilon$.

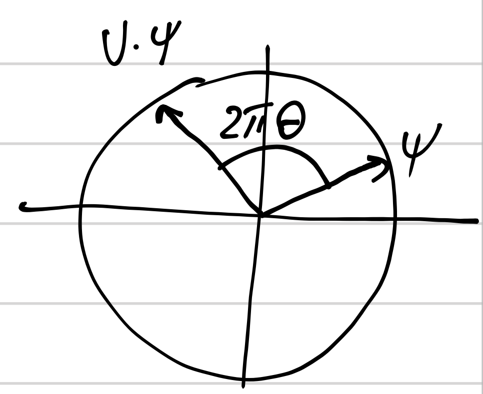
note: e^{-iH} is unitary. U_t runs e^{-iH} t times.

Fact: s-sparse H can be eff. sim.

[using Trotter-Suzuki method, when $s = \mathrm{poly}(n)$]

(2) Phase estimation: $U \in \mathbb{C}^{N \times N}$ unitary.

Let $|\psi\rangle \in \mathbb{C}^{N}$ be an eigenvector of U.



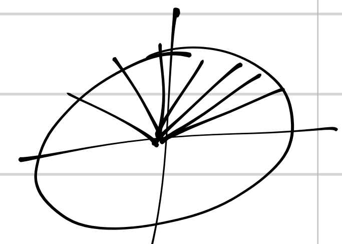
$$\Rightarrow \quad U \cdot |\psi\rangle = e^{2\pi i \cdot \theta} \cdot |\psi\rangle$$

phase estimation:

$$|\psi\rangle \longrightarrow |\psi\rangle |\overset{\ell\text{-bits}}{\tilde{\theta}}\rangle \quad \text{where} \quad |\theta - \tilde{\theta}| < 1/2^{\ell}$$

can be done with $O(2^{\ell})$ applications of U.
Nothing else known about U!!

How? QFT (Kitaev '96)



$$|0^\ell\rangle|\psi\rangle \xrightarrow{H^{\otimes \ell}} \sum_{j=0}^{2^\ell-1}|j\rangle|\psi\rangle \xrightarrow[\text{of } U]{2^\ell \text{ applications}}$$

$$\sum_{j=0}^{2^\ell-1}|j\rangle\left(U^j|\psi\rangle\right) = \sum_{j=0}^{2^\ell-1} e^{2\pi i\cdot j\theta}|j\rangle|\psi\rangle \xrightarrow[\text{left cell}]{QFT_\ell^{-1}} \big|\underbrace{2^\ell\cdot\tilde{\theta}}_{\ell\text{-bits}}\big\rangle|\psi\rangle$$

$$|\theta-\tilde{\theta}|<1/2^\ell$$

More generally: Let $U=\sum_{j=1}^{N} \underbrace{e^{2\pi i\cdot\theta_j}}_{\text{eigenvalues of } U.}\cdot|u_j\rangle\langle u_j|$

Let $|\psi\rangle = \sum_{j=1}^{N}\psi_j\cdot|u_j\rangle$

Phase estimation thm: ℓ bits, $|\theta_j-\tilde{\theta}_j|<\frac{1}{2^\ell}$

$$|\psi\rangle \longrightarrow \sum_{j=1}^{N}\psi_j\cdot|u_j\rangle|\tilde{\theta}_j\rangle$$

can be done by applying U 2^ℓ times

Nothing else needed about U!!

So: 2^{10} evals of U $\Rightarrow$ 10-bit eigenvalue accuracy.

e.g., for $U=e^{2\pi i\cdot H}$ where H is eff. simulatable.

Cute application: $F:\{0,1\}^n\to\{0,1\}$.
estimate $K := \#\{x\in\{0,1\}^n \mid F(x)=1\}$

Recall Grover: $|\psi\rangle\in\operatorname{span}(\bar{\alpha},\bar{\beta}) \Rightarrow R\cdot D\cdot|\psi\rangle$ rotated by $\theta \approx 2\sqrt{\frac{K}{2^n}}$ radians

$R\cdot D$ has two eigenvectors $u_1,u_2\in\operatorname{span}(\bar{\alpha}\ \bar{\beta})$

with eigenvalues $e^{2\pi i\theta}$, $e^{2\pi i(1-\theta)}$. Say $|\psi\rangle=\psi_1|u_1\rangle+\psi_2|u_2\rangle$.

$u_1 = i\bar{\alpha}+\bar{\beta}$

$u_2 = i\bar{\alpha}-\bar{\beta}$

$|\psi\rangle|0^\ell\rangle \xrightarrow[\text{est.}]{\text{phase}} \psi_1|u_1\rangle|\tilde\theta\rangle + \psi_2|u_2\rangle|1-\tilde\theta\rangle$

$\Rightarrow$ either right cell gives ℓ-bit approx. of $\theta \approx 2\sqrt{\frac{K}{2^n}} \Rightarrow$ approx. of K

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(3) controlled rotation: For ℓ-bit $\theta \in [0,1]$

$$|\theta\rangle|0\rangle \longrightarrow |\theta\rangle\left(\sqrt{1-\theta^2}\cdot|0\rangle + \theta\cdot|1\rangle\right)$$

can be implemented using $O(\ell)$ gates

Proof: First do $|\theta\rangle|0\rangle \rightarrow |\arccos(\theta)\rangle|0\rangle$
Then apply ℓ controlled R_w gates $(w = e^{2\pi i/2^\ell})$.

HHL (2009): $A \in \mathbb{C}^{N\times N}$ Hermitian, eff sim.

$$|b\rangle = \frac{1}{\|b\|}\sum_{j=1}^{N} b_j\cdot|j\rangle \in \mathbb{C}^N$$

want $\sum_{j=1}^{N} \tilde{x}_j\cdot|j\rangle$ where $\|\tilde{x} - A^{-1}b\| \le \frac{1}{2^\ell} \le \varepsilon$.

observation: Write $A = \sum_{j=1}^{N} \lambda_j\cdot|u_j\rangle\langle u_j|$

Then:
$$A^{-1} = \sum_{j=1}^{N}\frac{1}{\lambda_j}|u_j\rangle\langle u_j|, \qquad e^{2\pi i A} = \sum_{j=1}^{N} e^{2\pi i\lambda_j}|u_j\rangle\langle u_j|$$

assume: $\forall j: |\lambda_j| \ge 1$. $(|\lambda_{max}| \le K$ = cond. number$)$

write $|b\rangle = \sum_{j=1}^{N}\beta_j\cdot u_j \in \mathbb{C}^N$

Then:
$$A^{-1}\cdot b = \sum_{j=1}^{N}\frac{\beta_j}{\lambda_j}\cdot u_j \in \mathbb{C}^N.$$

The alg:

(1) run phase estimation on $|\vec{b}\rangle$ and operator $U := e^{2\pi i A}$:

$$|\vec{b}\rangle |0\rangle \xrightarrow[\text{est. on } U]{\text{phase}} \sum_{j=1}^{n} \beta_j \cdot |u_j\rangle \cdot |\tilde{\lambda}_j\rangle |0\rangle$$

bit

ℓ bits

(need A to be eff. sim.)

(2) controlled rotate of $|0\rangle$:

ℓ-bits

$$\xrightarrow[\text{computation}]{\text{classical}} \sum_{j=1}^{N} \beta_j \cdot |u_j\rangle \cdot |\tilde{\lambda}_j, \frac{1}{\tilde{\lambda}_j}\rangle |0\rangle$$

$(|\tilde{\lambda}_j| \geq 1)$

$$\xrightarrow[\text{rotate of } |0\rangle]{\text{controlled}}$$

$$\sum_{j=1}^{N} \beta_j \cdot |u_j\rangle \cdot |\tilde{\lambda}_j, \frac{1}{\tilde{\lambda}_j}\rangle \left(\sqrt{1-\frac{1}{\tilde{\lambda}_j^2}}\,|0\rangle + \frac{1}{\tilde{\lambda}_j}|1\rangle\right)$$

$$\xrightarrow[\text{right bit until get 1}]{\text{measure}} \sum_{j=1}^{N} \frac{\beta_j}{\tilde{\lambda}_j} \cdot |u_j\rangle \cdot |\tilde{\lambda}_j, \frac{1}{\tilde{\lambda}_j}\rangle \xrightarrow[\text{phase est.}]{\text{uncompute}}$$

$$\longrightarrow \sum_{j=1}^{N} \frac{\beta_j}{\tilde{\lambda}_j} \cdot |u_j\rangle = \sum_{j=1}^{N} \tilde{x}_j \cdot |j\rangle$$

$$A^{-1}\vec{b} = \sum_{j=1}^{N} \frac{\beta_j}{\lambda_j} \cdot u_j$$

Time until measure $|1\rangle$ depends on size of $|\lambda_{max}| < \kappa$, as does $\|x - \tilde{x}\|_2$.